

A CONNECTED  $F$ -SPACE

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ABSTRACT. We present an example of a compact connected  $F$ -space with a continuous real-valued function  $f$  for which the set  $\Omega_f = \bigcup \{\text{Int } f^{\leftarrow}(x) : x \in \mathbb{R}\}$  is not dense. This indirectly answers a question from Abramovich and Kitover in the negative.

## INTRODUCTION

The purpose of this note is to give a positive answer to Problem 4 from [1]. The problem asks whether there are a compact and connected  $F$ -space  $K$  and a continuous real-valued function  $f$  on  $K$  such that the set  $\Omega_f$  is not dense in  $K$ , where  $\Omega_f = \bigcup \{\text{Int } f^{\leftarrow}(x) : x \in \mathbb{R}\}$ . If  $K$  is such a space then the vector lattice  $C(K)$  has a maximal  $d$ -independent system that is not a  $d$ -base, which answers Problem 1 from the same paper in the negative.

As defined in [1] a  $d$ -independent system in a vector lattice  $X$  is a subset  $D$  with the property that for every band  $B$  in  $X$ , for every finite subset  $F$  of  $D$  and every choice  $\{c_d : d \in F\}$  of non-zero scalars the condition  $\sum_{d \in F} c_d d \perp B$  implies  $d \perp B$  for all  $d \in F$ . A  $d$ -independent system  $D$  is a  $d$ -basis if for every  $x \in X$  one can find a full system  $\mathcal{B}$  of pairwise disjoint bands and a subset  $\{y_B : B \in \mathcal{B}\}$  of  $X$  such that for each  $B$  the element  $y_B$  is a linear combination of members of  $D$  and  $x - y_B \perp B$ .

In topological terms a  $d$ -independent system in  $C(K)$  is a subset  $D$  such that for every nonempty open subset  $O$  the family of nonzero members of  $\{d \upharpoonright O : d \in D\}$  is linearly independent. The  $d$ -independent set  $D$  is a  $d$ -basis if for each  $g \in C(K)$  there is a pairwise disjoint family  $\mathcal{O}$  of open sets with a dense union and such that for every  $g \in C(K)$  and every  $O \in \mathcal{O}$  the restriction  $g \upharpoonright O$  is a linear combination of finitely members of  $\{d \upharpoonright O : d \in D\}$ .

As observed in [1] for our example  $K$  the set  $\{1\}$ , consisting of just the constant function with value 1, is maximally  $d$ -indepent in  $C(K)$ . Indeed, if  $g$  is not constant then its image  $g[K]$  is a nontrivial interval; we let  $t$  be its mid-point. Because  $K$  is an  $F$ -space the closed sets  $\text{cl } g^{\leftarrow}((-\infty, t])$  and  $\text{cl } g^{\leftarrow}([t, \infty))$  are disjoint and because  $K$  is connected they do not cover  $K$ . The nonempty open set  $\text{Int } g^{\leftarrow}(t)$  now witnesses that  $\{1, g\}$  is not  $d$ -independent. The continuous function  $f$ , on the other hand, witnesses that  $\{1\}$  is not a  $d$ -basis, for clearly any ‘ $d$ -linear combination’  $g$  of  $\{1\}$  must have its set  $\Omega_g$  dense in  $K$ .

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## 1. THE EXAMPLE

Let  $S$  be the unit square, i.e.,  $S = [0, 1]^2$ . We consider the product  $\mathbf{S} = \omega \times S$ , its Čech-Stone compactification  $\beta\mathbf{S}$  and the extension  $\beta\pi$  of the map  $\pi : \mathbf{S} \rightarrow \omega$ , defined by  $\pi(n, x) = n$ .

For each free ultrafilter  $u \in \beta\omega \setminus \omega$  the fiber  $S_u = \beta\pi^{-1}(u)$  is a continuum — see, e.g., [2]. As it is a closed subset of the Čech-Stone remainder  $\mathbf{S}^*$  it is also a compact  $F$ -space.

The function  $f : \mathbf{S} \rightarrow [0, 1]$ , defined by  $f(n, x, y) = x$  is clearly continuous; we write  $f_u$  for the restriction of  $\beta f$  to  $S_u$ . We shall find a continuum  $K$  in  $S_u$  such that  $g = f_u \upharpoonright K$  is as required, i.e.,  $\Omega_g$  is not dense in  $K$ .

We need to describe the boundaries of the fibers of  $f$ . We define  $L_t = f_u^{-1}(t) \cap \text{cl } f_u^{-1}([0, t])$  and  $R_t = f_u^{-1}(t) \cap \text{cl } f_u^{-1}((t, 1])$ ; note that  $L_0 = R_1 = \emptyset$ .

**Lemma 1.1.** *For each  $t \in (0, 1)$  the sets  $L_t$  and  $R_t$  are exactly the components of the boundary  $\text{Fr } f_u^{-1}(t)$  of  $f_u^{-1}(t)$ .*

*Proof.* Because  $S_u$  is an  $F$ -space the closed sets  $L_t$  and  $R_t$  are disjoint; they cover  $\text{Fr } f_u^{-1}(t)$  and, because  $S_u$  is connected, both are nonempty. This shows that  $\text{Fr } f_u^{-1}(t)$  has at least two components.

To finish we show that  $L_t$  and  $R_t$  are connected. For this we first observe that the ‘rectangle’  $P_{s,r} = S_u \cap \text{cl}(\omega \times [s, r] \times [0, 1])$  is connected whenever  $s < r$ . This in turn implies that  $L_{s,t} = \text{cl} \bigcup_{s < r < t} P_{s,r}$  is connected whenever  $s < t$ . It is readily verified that  $L_t = \bigcap_{s < t} L_{s,t}$ , hence  $L_t$  is connected as the intersection of a chain of continua. By symmetry  $R_t$  is also connected.  $\square$

This argument also shows that  $R_0 = \text{Fr } f_u^{-1}(0)$  and  $L_1 = \text{Fr } f_u^{-1}(1)$  are connected.

We need some more notation. We denote by  $B_u$  the intersection of  $S_u$  with the closure, in  $\beta\mathbf{S}$ , of  $\omega \times [0, 1] \times \{0\}$  — the bottom line of  $S_u$  — and likewise the top line  $T_u$  is  $S_u \cap \text{cl}_{\beta\mathbf{S}}(\omega \times [0, 1] \times \{1\})$ . The continuum  $K$  will be defined as the union of the bottom line of  $S_u$  and a family of vertical continua, each of which meet both the bottom and top lines.

To define this family we define sequences  $\langle X_\alpha \rangle_\alpha$  and  $\langle f_\alpha \rangle_\alpha$  of closed sets and functions respectively, by recursion. To begin let  $X_0 = S_u$ . Given  $X_\alpha$  put  $f_\alpha = f_u \upharpoonright X_\alpha$  and define  $X_{\alpha+1} = X_\alpha \setminus \bigcup_t \text{Int}_\alpha f_\alpha^{-1}(t)$ , where  $\text{Int}_\alpha$  is the interior operator in  $X_\alpha$ . If  $\alpha$  is a limit we just let  $X_\alpha = \bigcap_{\beta < \alpha} X_\beta$ .

**Lemma 1.2.** *For every  $\alpha$  and every  $t$  the intersections  $X_\alpha \cap L_t$  and  $X_\alpha \cap R_t$  are nonempty*

*Proof.* The proof is by induction on  $\alpha$ .

The statement is clearly true for  $\alpha = 0$  and the case  $\alpha = 1$  is covered by Lemma 1.1, whose proof also establishes the successor step in the induction. Indeed, to show that  $X_{\alpha+1} \cap L_t \neq \emptyset$  we note that, by the inductive assumption we know that  $P_{s,r} \cap X_\alpha$  meets  $L_q$  and  $R_q$ , whenever  $s < q < r$ . Therefore  $L_{s,t} \cap X_\alpha \neq \emptyset$  for all  $s < t$ ; using compactness we find that  $L_t \cap X_{\alpha+1} = \bigcap_{s < t} (L_{s,t} \cap X_\alpha)$  is nonempty.

The case of limit  $\alpha$  follows using compactness as well.  $\square$

**Lemma 1.3.** *Every component of  $X_\alpha$  meets both  $B_u$  and  $T_u$ .*

*Proof.* This is clear when  $\alpha = 0$  and as in the previous lemma we draw inspiration from the proof of Lemma 1.1 for the argument in the successor step. Observe first

that a component of  $X_{\alpha+1}$  is necessarily a subset of some  $L_t$  or  $R_t$ : these sets are the components of  $X_1$ .

Let  $C$  be a component of  $L_t$  and let  $O$  be an arbitrary clopen neighbourhood of  $C$  in  $L_t \cap X_{\alpha+1}$ ; choose open sets  $U$  and  $V$  in  $S_u$  with disjoint closures such that  $O \subseteq U$  and  $(L_t \cap X_{\alpha+1}) \setminus O \subseteq V$ . There is an  $s$  such that  $L_{s,t} \cap X_\alpha \subseteq U \cup V$ . Choose  $r \in (s, t)$  such that some component,  $D$ , of  $X_\alpha \cap (L_r \cup R_r)$  meets  $U$ ; then  $D \subseteq U$  and it follows that  $U$  intersects both  $B_u$  and  $T_u$ . Because  $O$  and  $U$  were arbitrary it follows that  $C$  must meet  $B_u$  and  $T_u$  as well.

In case  $\alpha$  is a limit and  $C$  a component we have  $C = \bigcap_{\beta < \alpha} C_\beta$ , where  $C_\beta$  is the component of  $X_\beta$  that contains  $C$ ; the  $C_\beta$ 's form a chain and all of them intersect  $B_u$  and  $T_u$  and hence by compactness so does  $C$ .  $\square$

There will be a minimal ordinal  $\delta$  such that  $X_\delta = X_{\delta+1}$  (some information on  $\delta$  will be given in the next section). This means that  $\text{Int}_\delta f_\delta^-(t) = \emptyset$  for all  $t$ .

Our continuum  $K$  is the union of  $B_u$  and  $X_\delta$ . Because all components of  $X_\delta$  meet  $B_u$  we know that  $K$  is indeed connected. Because each component meets  $T_u$  we know that  $K$  reaches all the way up to  $T_u$ ; by the choice of  $\delta$  we get that  $\text{Int}_K g^-(t) \subseteq B_u$  for all  $t$ . Thus  $\Omega_g \subseteq B_u$  and the latter set is certainly not dense in  $K$ .

## 2. A REMARK AND A QUESTION

The first (and erroneous) version of  $K$  was simply  $B_u \cup \bigcup_{0 < t \leq 1} R_t \cup \bigcup_{0 \leq t < 1} L_t$ . After I realized that the restriction of  $f$  to this subspace did not provide an example it became clear that the procedure of removing interiors of fibers had to be iterated, which lead to the sequence  $\langle X_\alpha \rangle_\alpha$ . We can provide some information on the ordinal  $\delta$  at which the sequence becomes constant.

**Proposition 2.1.**  $\delta < \mathfrak{c}^+$

*Proof.* Let  $\mathcal{B}$  be a base for  $S_u$  of cardinality  $\mathfrak{c}$ . For every  $\alpha < \delta$  there is a  $B_\alpha \in \mathcal{B}$  such that  $\emptyset \neq B_\alpha \cap X_\alpha \subseteq X_\alpha \setminus X_{\alpha+1}$ . Clearly  $\alpha \mapsto B_\alpha$  is one-to-one, which establishes that  $|\delta| \leq \mathfrak{c}$ .  $\square$

The  $F$ -space property implies that  $\delta$  cannot be a successor ordinal, nor an ordinal of countable cofinality.

**Lemma 2.2.** *If  $\alpha < \delta$  then  $X_\alpha \setminus X_{\alpha+1}$  meets every  $L_t$  and every  $R_t$ .*

*Proof.* This is basically a consequence of the homogeneity of the unit interval. If  $h : [0, 1] \rightarrow [0, 1]$  is a homeomorphism then it induces an autohomeomorphism  $h_u$  of  $S_u$  via the map  $(n, x, y) \mapsto (n, h(x), y)$  from  $\mathbf{S}$  to itself. The map  $h_u$  simply permutes the fibers  $f^-(t)$  and it is relatively straightforward to show by induction that  $h_u[X_\alpha] = X_\alpha$  for all  $\alpha$ . There are enough maps  $h$  to ensure that once  $X_\alpha \setminus X_{\alpha+1}$  meets one  $L_t$  (or one  $R_t$ ) it meets all  $L_s$  and all  $R_s$ .  $\square$

**Proposition 2.3.**  $\delta$  is not a successor ordinal.

*Proof.* Let  $\alpha < \delta$ , we show that  $\alpha + 1 < \delta$ . Fix  $t \in (0, 1)$  and let  $\langle t_n \rangle_n$  be a sequence in  $[0, 1]$  that converges to  $t$  from above. By Lemma 2.2 we can pick  $x_n \in L_{t_n} \cap X_\alpha \setminus X_{\alpha+1}$  for each  $n$ .

Clearly every point in the closure of  $\{x_n\}_n$  belongs to  $X_{\alpha+1}$ ; we show that none belong to  $X_{\alpha+2}$ . To see this observe that the  $F_\sigma$ -sets  $F = \{x_n\}_n$  and  $G = f^-[t, 1]$

are *separated* in  $S_u$ , i.e.,  $\text{cl } F \cap G = \emptyset = F \cap \text{cl } G$ . Using normality in the form of Urysohn's Lemma one can find a continuous function  $h : S_u \rightarrow [-1, 1]$  such that  $h[F] \subseteq [-1, 0)$  and  $h[G] \subseteq (0, 1]$ . But now the  $F$ -space property applies to show that  $\text{cl } F \cap \text{cl } G = \emptyset$ .  $\square$

In a similar way we can prove the following.

**Proposition 2.4.** *The ordinal  $\delta$  has uncountable cofinality.*

*Proof.* We choose an increasing sequence  $\langle \alpha_n \rangle_n$  of ordinals below  $\delta$ ; we show that  $\lim_n \alpha_n < \delta$ .

As in the previous proof we fix  $t \in (0, 1)$  and a sequence  $\langle t_n \rangle_n$  converging to  $t$  from above. As before we choose  $x_n \in L_{t_n} \cap X_{\alpha_n} \setminus X_{\alpha_n+1}$  for all  $n$ .

As in the previous proof the  $F$ -space property now ensures that every point in the closure of  $\{x_n\}_n$  belongs to  $X_\alpha \setminus X_{\alpha+1}$ .  $\square$

We deduce that  $\delta$  must be at least  $\omega_1$  but the following question remains.

**Question 1.** What is the exact value of  $\delta$ ?

#### REFERENCES

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